

# Noether symmetry in $F(T)$ gravity with $f$ -essence

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## Abstract

In  $F(T)$  gravity theory, a Friedman-Robertson-Walker cosmological model with  $f$ -essence where fermion field is non-minimally coupled with the gravitational field is studied. Using the Noether symmetry approach the possible forms of  $F(T)$  gravity and the non-canonical fermionic lagrangian  $K$  are determined. Cosmological solutions of the condered models describing the accelerated and decelerated periods of the universe are found.

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## 1 Introduction

A new challenge for modern cosmology is to build some theoretical models to explain the dynamics of the large scale Universe. Different cosmological data supports the idea of an accelerating Universe [1, 2]. There are two approaches as attempts to resolve this problem: One is to keep general relativity (GR) and add some exotic matter field(s), called dark energy (DE) [3, 4, 5]. The most viable candidates for the role of dark energy are some kinds of the scalar, fermion or tachyon fields as quintessence [6, 7, 8], phantom energy [9, 10],  $k$ -essence [11, 12],  $f$ -essence [14]-[16] and  $g$ -essence [17]-[19]. However, this approach leads to some additional issues, such as the existence of negative entropy, future singularity energy condition violation and etc. Furthermore in this approach, all types of DEs, violate energy conditions as fundamental blocks of the GR. The second approach is called modified theories of gravity, and originally proposed by Buchdahl in [21], when he generalized GR to a nonlinear form of Lagrangian with an Ricci scalar  $R$  on  $F(R)$  gravity (see [22, 23] for a revisit). Also, it was recently proposed another new modified theory of gravity so-called  $F(T)$  gravity [24] -[51], which is a generalization teleparallel gravity proposed by Einstein. In this theory is used the Weitzenböck connection, then as in GR used the Levi-Civita connection. Next generalization of  $F(R)$  and  $F(T)$  gravity theory is  $F(R, T)$  gravity, with different

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meanings for  $T$ , the one for trace of the energy-momentum tensor of and its higher derivative extensions [52]-[57].

Symmetry is an essential feature to build any theory of particle physics and gravity. If we find the point like Lagrangian, we can study the family of symmetries which are corresponding to the conserved quantities. These types of the symmetries are called Noether symmetry. The conserved quantity which is associated to the symmetry generator is called as Noether charge, in correspondence to the same terminology in particle physics and gauge theories.

In recent years Noether symmetry successfully used for the construction of various cosmological models [58]-[62]. Noether symmetry has been used to investigate cosmology in phantom quintessence cosmology, for non-minimally coupled fermion fields, for boson and fermion field, in  $F(R)$  cosmology, in teleparallel gravity, in  $F(T)$  gravity, in  $F(R, T)$  gravity and Noether symmetry in quantum cosmology [63]-[79].

In this paper, we consider Noether symmetry for  $F(T)$  gravity with f-essence, as a framework to explain the present accelerated expansion of the Universe. The matter component is assumed to be a generic function of the kinetic energy of the fermion field. To find the explicit form of the function  $F(T)$ , we utilize generalized Noether theorem and use generalized vector fields as infinitesimal generators of the symmetries for the corresponding Lagrangian. We study the cosmological consequences of the obtained results.

This paper is organized as follows: in Sec. 2, we review the basics of  $f(T)$  gravity. In Sec. 3, we derive the equations of motions for  $F(T)$  gravity with  $f$ -essence with non-minimally coupled to the gravitational field from a point-like Lagrangian in a spatially flat Friedman-Robertson-Walker metric (FRW). In Sec. 4, we look for the existence of Noether symmetry for the point like Lagrangian is the subject, where the possible forms of the coupling and of the potential density are determined. In Sec. 5, the field equations are solved for couplings and potential densities found in the previous sections and the cosmological solutions are investigated. The final remarks and conclusions are the subject of the last section.

We adopted the signature as  $(+, -, -, -)$  and natural units  $8\pi G = c = \hbar = 1$ .

## 2 $F(T)$ gravity

The action for  $F(T)$  gravity is given by

$$S = \int d^4x e [F(T) + L_m], \quad (1)$$

where  $e = \det(e_\mu^i) = \sqrt{-g}$ ,  $T$  is the torsion scalar and  $L_m$  stands for the matter Lagrangian.

The torsion scalar is defined by the expression,

$$T = S^{\mu\nu}_\rho T^\rho_{\mu\nu}, \quad (2)$$

where  $S^{\mu\nu}_\rho$  is given as follows

$$S^{\mu\nu}_\rho = \frac{1}{2} \left( K^\mu_\rho{}^{\nu\theta} + \delta^\mu_\rho T^{\theta\nu}_\theta - \delta^\nu_\rho T^{\theta\mu}_\theta \right), \quad (3)$$

and torsion tensor  $T^\rho_{\mu\nu}$  is given as

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} = e^\rho_i (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu). \quad (4)$$

Here  $e^i_\mu$  are the components of the non-trivial tetrad field  $e_i$  in the coordinate basis. There is a wide extra choiceto find a good tetrad basis. The reason backs to the following identity regsrding metric in vielbein formalism:

$$g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu, \quad (5)$$

where  $\eta_{ij} = \text{diag}(1, -1, -1, -1)$  is the Minkowski metric for the tangent space. Consequently, with a given metric, there exist infinite set of basis tetrad fields  $e_\mu^i$  all can satisfy the following properties:

$$e_\mu^i e_j^\mu = \delta_j^i, \quad e_\mu^i e_i^\nu = \delta_\mu^\nu. \quad (6)$$

The procedure for evaluating the tetrad field has been studied in literature by many authors but so far, there is not fixed manifest to find a good strategy to exclude bad tetrads. Notice that the Latin alphabets ( $i, j, \mu, \nu = 0, 1, 2, 3$ ) will be used to denote the tangent space indices and the Greek alphabets ( $\mu, \nu, \dots = 0, 1, 2, 3$ ) to denote the spacetime indices. The contorsion tensor  $K_\rho^{\mu\nu}$  is defined as

$$K_\rho^{\mu\nu} = -\frac{1}{2} (T_\rho^{\mu\nu} - T_\rho^{\nu\mu} - T_\rho^{\mu\nu}), \quad (7)$$

which is equal to the difference between a pair of Weitzenböck connections. The variation of action (1) with respect to the vierbein field leads to the following field equations,

$$\left[ e^{-1} \partial_\mu (e S_i^\mu \nu) - e_i^\lambda T_{\mu\lambda}^\rho S_\rho^{\nu\mu} \right] F_T + S_i^{\mu\nu} \partial_\mu (T) F_{TT} + \frac{1}{4} e_i^\nu F = \frac{1}{2} k^2 H_i^\rho T_\rho^\nu. \quad (8)$$

Here  $T_{\mu\nu}$  is the energy-momentum tensor given as

$$T_\rho^\nu = \text{diag}(\rho_m, -p, -p, -p), \quad (9)$$

where  $\rho_m$  and  $p_m$  are the density and pressure of matter fields inside the Universe.

### 3 $F(T)$ gravity with $f$ -essence

In this section, we derived the equations of motions for  $F(T)$  gravity with  $f$ -essence with non-minimally coupling to the gravitational background. The gravitational action for  $f$ -essence is given by the following expression:

$$S = \int d^4x e [F(T) + 2K(Y, u)], \quad (10)$$

where  $K$  is some function of its arguments,  $u = \bar{\psi}\psi$ ,  $\psi = (\psi_0, \psi_1, \psi_2, \psi_3)^T$  is a fermionic function and  $\bar{\psi} = \psi^\dagger \gamma^0$  denotes its adjoint function. Furthermore, the kinetic part is defined by the following:

$$Y = 0.5i [\bar{\psi} \Gamma^\mu D_\mu \psi - (D_\mu \bar{\psi}) \Gamma^\mu \psi], \quad (11)$$

Here the differential operator  $D_\mu$  is the covariant derivative and  $\Gamma^\mu = e_a^\mu \gamma^a$ .

We will consider here the simplest homogeneous and isotropic cosmological model, with the following metric:

$$ds^2 = dt^2 - a^2(t) [dx^2 + dy^2 + dz^2], \quad (12)$$

where  $a(t)$  is the scale factor of the Universe. For this metric, the good vierbein basis found to be  $(e_a^\mu) = \text{diag}(1, 1/a, 1/a, 1/a)$  and  $(e_\mu^a) = \text{diag}(1, a, a, a)$ .

The Dirac matrices of curved spacetime  $\Gamma^\mu$  are

$$\Gamma^0 = \gamma^0, \quad \Gamma^j = a^{-1} \gamma^j, \quad \Gamma^5 = -i\sqrt{-g} \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 = \gamma^5, \quad \Gamma_0 = \gamma^0, \quad \Gamma_j = a \gamma^j (j = 1, 2, 3). \quad (13)$$

Hence we get

$$\Omega_0 = 0, \quad \Omega_j = \frac{1}{2} \dot{a} \gamma^j \gamma^0 \quad (14)$$

The kinetic part is written as the following:

$$Y = \frac{1}{2} i (\bar{\psi} \gamma^0 \dot{\psi} - \dot{\bar{\psi}} \gamma^0 \psi). \quad (15)$$

Finally, we note that the gamma matrices are given as follow:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (16)$$

here  $I = \text{diag}(1, 1)$  and the  $\sigma^k$  are Pauli  $SU(2)$  matrices having the following form

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (17)$$

In fact selecting suitable Lagrange multipliers and integrating by parts to eliminate higher order derivatives, the Lagrangian  $\mathcal{L}$  transforms to the canonical form. In physical units, the action is

$$S = \int d^4x \left[ a^3 \left( h(u)F(T) - \lambda (T + 6H^2) + 2K(Y, u) \right) \right], \quad (18)$$

where  $u = \bar{\psi}\psi$  is the bilinear function. In order to determine  $\lambda$ , we have to vary the action with respect to  $T$ , that is

$$h \frac{dF}{dT} \delta T - \lambda \delta T = 0, \quad (19)$$

in which we obtain:

$$\lambda = hF_T. \quad (20)$$

where  $\lambda$  is a Lagrange multiplier.

Therefore, the action (18) can be rewritten as

$$S = \int d^4x \left[ a^3 \left( hF - hF_T (T + 6H^2) + 2K \right) \right], \quad (21)$$

and then the point-like Lagrangian reads

$$L = a^3 hF - a^3 hF_T T - 6a\dot{a}^2 hF_T + 2a^3 K. \quad (22)$$

It is well known that, for a dynamical system, the Euler-Lagrange equation is defined by the following:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0, \quad (23)$$

where  $q_i$  are the generalized coordinates of the configuration space  $Q$ , and in our case  $q_i = a, \psi, \bar{\psi}$  and  $T$ . Substituting the Lagrangian (22) into the Euler-Lagrange equation (23), we obtain

$$F_{TT} (T + 6H^2) = 0, \quad (24)$$

$$4HF_{TT}\dot{T} + \left( 6H^2 + 4\dot{H} - T + 4H\frac{\dot{h}}{h} \right) F_T + F + \frac{2}{h}K = 0, \quad (25)$$

$$K_Y \dot{\psi} + 0.5 \left( 3HK_Y + \dot{K}_Y \right) \psi - iK_u \gamma^0 \psi - 0.5i \left( F - F_T T - 6H^2 F_T \right) h_u \gamma^0 \psi = 0, \quad (26)$$

$$K_Y \dot{\bar{\psi}} + 0.5 \left( 3HK_Y + \dot{K}_Y \right) \bar{\psi} + iK_u \bar{\psi} \gamma^0 + 0.5i \left( F - F_T T - 6H^2 F_T \right) h_u \bar{\psi} \gamma^0 = 0, \quad (27)$$

where  $H = \frac{\dot{a}}{a}$  denotes the Hubble parameter. From equations (24) one can be seen  $T = -6H^2$ , which corresponds to the basic consideration of model.

We also consider the energy condition

$$E = \frac{\partial L}{\partial \dot{a}} \dot{a} + \frac{\partial L}{\partial \dot{T}} \dot{T} + \frac{\partial L}{\partial \dot{\psi}} \dot{\psi} + \frac{\partial L}{\partial \dot{\bar{\psi}}} \dot{\bar{\psi}} - L, \quad (28)$$

Here the dot indicates the derivatives with respect to the cosmic time  $t$ . By combining the equations (22) and (28), we obtain

$$(F_T T - 6H^2 F_T - F) h + 2(Y K_Y - K) = 0. \quad (29)$$

So finally, we have the following system of equations for our model:

$$12H^2 F_T + F - \frac{2}{h} \rho_f = 0, \quad (30)$$

$$48H^2 \dot{H} F_{TT} - 4 \left( 3H^2 + \dot{H} + H \frac{\dot{h}}{h} \right) F_T - F - \frac{2}{h} p_f = 0, \quad (31)$$

$$K_Y \dot{\psi} + 0.5 \left( 3H K_Y + \dot{K}_Y \right) \psi - i K_u \gamma^0 \psi - 0.5i (F - F_T T - 6H^2 F_T) h_u \gamma^0 \psi = 0, \quad (32)$$

$$K_Y \dot{\bar{\psi}} + 0.5 \left( 3H K_Y + \dot{K}_Y \right) \bar{\psi} + i K_u \bar{\psi} \gamma^0 + 0.5i (F - F_T T - 6H^2 F_T) h_u \bar{\psi} \gamma^0 = 0. \quad (33)$$

where

$$\rho_f = Y K_Y - K, \quad p_f = K \quad (34)$$

are the energy density and pressure of the fermionic field. From the equations (30)-(33), we see that these equations are nonlinear differential equations, respectively are difficult to find they solutions. To solve these equations need to find a form of the function  $F(T)$ ,  $h(u)$  and  $K(Y, u)$ . In the next section we will use Noether symmetry approach for find these function.

## 4 The Noether Symmetries Approach

The basic idea of Noether symmetry approach is that to find a class of symmetry generators  $\mathbf{X}$  with those generatr, the Lie derivative of the Lagrangian vanishes, i.e.

$$\mathcal{L}_{\mathbf{X}} = 0. \quad (35)$$

Our plan is to look for a possible set of the Noether symmetries for our model in terms of the components of the spinor field  $\psi = (\psi_0, \psi_1, \psi_2, \psi_3)^T$  and its adjoint  $\bar{\psi} = (\psi_0^\dagger, \psi_1^\dagger, -\psi_2^\dagger, -\psi_3^\dagger)$ . The existence of Noether symmetry given by the equation (35) implies the existence of a vector field  $\mathbf{X}$  such that

$$\mathbf{X} = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial T} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{T}} + \sum_{j=0}^3 \left( \eta_j \frac{\partial}{\partial \psi_j} + \dot{\eta}_j \frac{\partial}{\partial \dot{\psi}_j} + \chi_j \frac{\partial}{\partial \psi_j^\dagger} + \dot{\chi}_j \frac{\partial}{\partial \dot{\psi}_j^\dagger} \right), \quad (36)$$

where  $\alpha, \beta, \eta_i$  and  $\chi_i$  are depend on  $a, T, \psi_i$  and  $\psi_i^\dagger$  and their derivatives are determined from the following equations

$$\dot{\alpha} = \frac{\partial \alpha}{\partial a} \dot{a} + \frac{\partial \alpha}{\partial T} \dot{T} + \sum_{i=0}^3 \left( \frac{\partial \alpha}{\partial \psi_i} \dot{\psi}_i + \frac{\partial \alpha}{\partial \psi_i^\dagger} \dot{\psi}_i^\dagger \right), \quad (37)$$

$$\dot{\beta} = \frac{\partial \beta}{\partial a} \dot{a} + \frac{\partial \beta}{\partial T} \dot{T} + \sum_{i=0}^3 \left( \frac{\partial \beta}{\partial \psi_i} \dot{\psi}_i + \frac{\partial \beta}{\partial \psi_i^\dagger} \dot{\psi}_i^\dagger \right), \quad (38)$$

$$\dot{\eta}_j = \frac{\partial \eta_j}{\partial a} \dot{a} + \frac{\partial \eta_j}{\partial T} \dot{T} + \sum_{i=0}^3 \left( \frac{\partial \eta_j}{\partial \psi_i} \dot{\psi}_i + \frac{\partial \eta_j}{\partial \psi_i^\dagger} \dot{\psi}_i^\dagger \right), \quad (39)$$

$$\dot{\chi}_j = \frac{\partial \chi_j}{\partial a} \dot{a} + \frac{\partial \chi_j}{\partial T} \dot{T} + \sum_{i=0}^3 \left( \frac{\partial \chi_j}{\partial \psi_i} \dot{\psi}_i + \frac{\partial \chi_j}{\partial \psi_i^\dagger} \dot{\psi}_i^\dagger \right), \quad (40)$$

The condition (35) when applied to the Lagrangian (22) leads to an ploynomial expression of  $\dot{a}^2, \dot{T}^2, \dot{a}\dot{T}, \dot{a}\psi_i^\dagger, \dot{a}\psi_i, \dot{T}\psi_i^\dagger, \dot{T}\psi_i, \psi_i^\dagger\dot{\psi}_i, \dot{\psi}_i, \dot{a}, \dot{T}, \psi_i^\dagger$  and  $\dot{\psi}_i$ . Puuting the coefficients of the above expression set to the zero, one obtains the following system of coupled differential equations:

$$\alpha F_T + \beta a F_{TT} + 2a F_T \frac{\partial \alpha}{\partial a} + a F_T \frac{h_u}{h} \sum_{i=0}^3 \left( \epsilon_i \eta_i \psi_i^\dagger + \epsilon_i \chi_i \psi_i \right) = 0, \quad (41)$$

$$3\alpha (F - F_T T) - \beta a F_{TT} T + a \frac{h_u}{h} (F - F_T T) \sum_{i=0}^3 \left( \epsilon_i \eta_i \psi_i^\dagger + \epsilon_i \chi_i \psi_i \right) = 0, \quad (42)$$

$$3\alpha (K - Y K_Y) + a K_u \sum_{i=0}^3 \left( \epsilon_i \eta_i \psi_i^\dagger + \epsilon_i \chi_i \psi_i \right) = 0, \quad (43)$$

$$F_T \frac{\partial \alpha}{\partial T} = 0, \quad (44)$$

$$F_T \frac{\partial \alpha}{\partial \psi_i} = 0, \quad (45)$$

$$F_T \frac{\partial \alpha}{\partial \psi_i^\dagger} = 0, \quad (46)$$

$$\sum_{i=0}^3 \left( \frac{\partial \eta_i}{\partial a} \psi_i^\dagger - \frac{\partial \chi_i}{\partial a} \psi_i \right) = 0, \quad (47)$$

$$\sum_{i=0}^3 \left( \frac{\partial \eta_i}{\partial T} \psi_i^\dagger - \frac{\partial \chi_i}{\partial T} \psi_i \right) = 0, \quad (48)$$

$$3\alpha \psi_j + a \eta_i - a \sum_{i=0}^3 \left( \frac{\partial \eta_i}{\partial \psi_i^\dagger} \psi_i^\dagger - \frac{\partial \chi_i}{\partial \psi_i^\dagger} \psi_i \right) = 0, \quad (49)$$

$$3\alpha \psi_j^\dagger + a \chi_j + a \sum_{i=0}^3 \left( \frac{\partial \eta_i}{\partial \psi_j} \psi_i^\dagger - \frac{\partial \chi_i}{\partial \psi_j} \psi_i \right) = 0. \quad (50)$$

where  $\epsilon_i = 1$  for  $i = 1, 2$  and  $\epsilon_i = -1$  for  $i = 3, 4$ . From equations (43)-(45) we can see that the function  $\alpha$  dependent only a function of  $a$ . We can rewrite equation (42) as

$$\frac{3\alpha}{a K_u} (K - Y K_Y) = - \sum_{i=0}^3 \left( \epsilon_i \eta_i \psi_i^\dagger + \epsilon_i \chi_i \psi_i \right). \quad (51)$$

If we put this equation into equations (40)-(41), then the corresponding results are

$$\alpha F_T + \beta a F_{TT} + 2a F_T \frac{\partial \alpha}{\partial a} - \frac{3\alpha F_T h_u}{h K_u} (K - Y K_Y) = 0, \quad (52)$$

$$3\alpha (F - F_T T) - \beta a F_{TT} T - \frac{3\alpha h_u}{h K_u} (F - F_T T) (K - Y K_Y) = 0, \quad (53)$$

The equation (52), we can rewrite as

$$\beta a F_{TT} = \frac{3\alpha}{T} (F - F_T T) \left[ 1 - \frac{h_u}{h K_u} (K - Y K_Y) \right], \quad (54)$$

Put this equation in equation (51) and after some algebraic calculations, we have the following equation:

$$\frac{a}{\alpha} \frac{\partial \alpha}{\partial a} = 1 - \frac{3F}{2TF_T} \left[ 1 - \frac{h_u}{h K_u} (K - Y K_Y) \right] = n, \quad (55)$$

where  $n$  is a constant. Then, we find generator  $\alpha$  as

$$\alpha = \alpha_0 a^n. \quad (56)$$

where  $\alpha_0$  is a constant of integration. Now from the equation (54) and considering that  $h$  and  $K$  depend only on a function  $u$  and the function  $F$  is dependent of  $T$ , we have the following equation

$$\frac{h_u}{hK_u} (K - YK_Y) = 1 + \frac{2(n-1)TF_T}{3F} = m, \quad (57)$$

where  $n$  is a constant. From this equation, we have the following pair of ordinary differential equations

$$2(n-1)TF_T - 3(m-1)F = 0, \quad (58)$$

$$\frac{K - YK_Y}{K_u} = m \frac{h}{h_u}. \quad (59)$$

From the first equation we determine the form of the function  $F$  and its derivatives as

$$F = C_1 T^{\frac{3(m-1)}{2(n-1)}}. \quad (60)$$

Also, from the equations (48), (49) and (50), we find the solutions for the generators  $\eta_j$  and  $\chi_j$  as

$$\eta_j = - \left( \frac{3}{2} \alpha_0 a^{n-1} + \epsilon_j \eta_0 \right) \psi_j, \quad (61)$$

$$\chi_j = - \left( \frac{3}{2} \alpha_0 a^{n-1} - \epsilon_j \eta_0 \right) \psi_j^\dagger. \quad (62)$$

Substituting these values into equation (50) and using the  $\alpha$  given in (55), we have the following equation

$$\frac{K - YK_Y}{K_u} = u, \quad (63)$$

or

$$K - YK_Y - uK_u = 0, \quad (64)$$

To solve this equation we need to make the change of variables

$$Y, u \rightarrow p = Y - \nu u, \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial p}, \quad \frac{\partial}{\partial u} = -\nu \frac{\partial}{\partial p}. \quad (65)$$

where  $\nu$  is an another constant. Then the partial differential equation (63) is transformed into an ordinary differential equation

$$K - p \frac{dK}{dp} = 0, \quad (66)$$

Then, we find  $K$  as

$$K = K_0 (Y - \nu u) \quad (67)$$

where  $K_0$  is a integrable constant.

If you compare the equation (58) with equation (66) will be available the following expression

$$h = h_0 u^m \quad (68)$$

Finally, we put the equations (55), (59), (66) and (68) into (53), we are obtained

$$\beta = 2\alpha_0 (n-1) a^{n-1} T. \quad (69)$$

Thus, we find the explicit non-zero solutions for the functions  $\alpha$ ,  $\beta$ ,  $\eta_i$ ,  $\chi_i$ ,  $F$ ,  $K$  and  $h$ . Therefore, for our model Noether symmetry vector  $\mathbf{X}$  exists.

## 5 Cosmological solutions

In previous section we obtained the form of Noether symmetry for our model. Now we're ready to solve equations of motion and to find exact cosmological solutions. As a first step, we need to substitute the solutions (59), (66) and (67) in the field equations (32), (33). Then, we have

$$\dot{u} + 3\frac{\dot{a}}{a}u = 0, \quad (70)$$

so that

$$u = \frac{u_0}{a^3} \quad (71)$$

where  $u_0$  is an integration constant.

In order to determine the function dependence of the scale factor of the time  $a(t)$ , we need to substitute the values of functions  $F(T)$ ,  $h(u)$  and  $K(Y, u)$  obtained from equations (59), (66) and (68) into equation (30), then we have

$$\dot{a} = a_0 a^n. \quad (72)$$

The solution of this differential equation is given by:

$$a(t) = [a_0 (1 - n) (t - C_2)]^{\frac{1}{1-n}}, \quad (73)$$

where  $C_2$  is a constant of integration,  $n \neq 1$  and the constant  $a_0$  is

$$a_0 = \frac{1}{\sqrt{-6}u_0^{\frac{1}{3}(n-1)}} \left[ \frac{2K_0\nu(n-1)}{h_0C_1(n+2-3m)} \right]^{\frac{n-1}{3(m-1)}}. \quad (74)$$

and we can see

$$u = u_0 [a_0 (1 - n) (t - C_2)]^{-\frac{1}{1-n}}. \quad (75)$$

We find that the Hubble parameter is

$$H = \frac{\dot{a}}{a} = \frac{1}{(1-n)(t-C_2)}. \quad (76)$$

The energy density and pressure for this model are

$$\rho = \frac{3}{(1-n)^2(t-C_2)^2}, \quad (77)$$

$$p = -\frac{2n+1}{(1-n)^2(t-C_2)^2}. \quad (78)$$

The equation of state parameter for our model can be define as

$$\omega = \frac{p}{\rho} = -\frac{1}{3}(1+2n). \quad (79)$$

As was shown earlier for our model, the constant  $n \neq 1$ . In our model, we consider a value  $n > 1$  then we have  $\omega < -1$  that this phase is the phantom phase and if  $n = 0$  we have  $\omega = -\frac{1}{3}$  is the quintessence phase.

The deceleration parameter for the fermionic field define as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -n, \quad (80)$$

From this example we can see that for  $n > 0$  of our universe can be the accelerated, for  $n < 0$  decelerating expansion.

When  $n = -\frac{1}{2}$ , we can see

$$\rho_{tot} = \frac{4}{3(t-C_2)^2}, \quad p = 0. \quad (81)$$

From this example we have a standard pressureless matter field. As a vital fact, we conclude that the fermionic field behaves as both the phantom and quintessence phase of the accelerating expanding universe.



## 6 Conclusions

In this paper we have considered the Noether symmetry approach for  $F(T)$  gravity with  $f$  essence. We used the Noether symmetry approach for to determine forms of the physical quantities as  $F = C_1 T^{\frac{3(m-1)}{2(n-1)}}$ ,  $K = K_0 (Y - \nu u)$  and  $h = h_0 u^m$ . Taking the derivative of the scale factor at the time, we can determine the type of the parameter as the Hubble. Next, we have found values for the energy and pressure for the fermion field. Finally, for the our model we obtained is the equation of state parameter as  $\omega = \frac{p}{\rho} = -\frac{1}{3}(1 + 2n)$ . From this equation we can see that is the constant  $n$  can take the value  $n > 1$  then we have  $\omega < -1$  that this phase is the phantom phase and if  $n = 0$  we have  $\omega = -\frac{1}{3}$  is the quintessence phase. As was shown earlier for our model, the constant  $n \neq 1$ . However, when  $n = 1$ , for the our model has no physical meaning. Also considered was the case when  $n = -\frac{1}{2}$ . For this case we have  $\rho_{tot} = \frac{4}{3(t-C_2)^2}$  and  $p = 0$ . We can see that this solution for the fermionic field gives us a standart pressureless matter field.

## References

- [1] Perlmutter S. et al. *Measurements of omega and lambda from 42 high-redshift supernovae*. The Astrophysical Journal, **517**, N2, 565-586 (1999). [arXiv:astro-ph/9812133]
- [2] Riess et al. *Observational evidence from supernovae for an accelerating universe and a cosmological constant*. The Astronomical Journal, **116**, N3, 1009-1038 (1998). [arXiv:astro-ph/9805201]
- [3] Peebles P.J.E., Ratra B. *The cosmological constant and dark energy*. Reviews of Modern Physics, **75**, 559-606 (2003). [arXiv:astro-ph/0207347]
- [4] Copeland E.J., Sami M., and Tsujikawa S. *Dynamics of dark energy*. International Journal of Modern Physics D, **15**, 1753 (2006). [arXiv:hep-th/0603057]
- [5] Amendola L., Tsujikawa S. *Dark energy: Theory and observations*. Cambridge University Press. 491 (2010).
- [6] Chiba T., Okabe T., Yamaguchi M. *Kinetically driven quintessence*. Physical Review D, **62**, N2, 3511 (2000). [arXiv:astro-ph/9912463]
- [7] Tsujikawa S. *Quintessence: A Review*. Classical and Quantum Gravity, **30**, 214003 (2013). [arXiv:1304.1961]
- [8] Khurshudyan M., Chubaryan E. and Pourhassan B. *Interacting Quintessence Models of Dark Energy*. International Journal of Theoretical Physics, **53**, 2370-2378 (2014). [arXiv:1402.2385]
- [9] Caldwell R.R. *A Phantom Menace? Cosmological consequences of a dark energy component with super-negative equation of state*. Physical Letters B, **545**, 23-27 (2002). [arXiv:astro-ph/9908168]
- [10] Nojiri S. and Odintsov S. D. *Inhomogeneous equation of state of the universe: Phantom era, future singularity and crossing the phantom barrier*. Physical Review D, **72**, 023003 (2005). [arXiv:hep-th/0505215]
- [11] Armendariz-Picon C., Damour T., Mukhanov V.F. *k-inflation*. Physical Letters B, **458**, N7, 209-218 (1999). [arXiv:hep-th/9904075]
- [12] Armendariz-Picon C., Mukhanov V.F., Steinhardt P.J. *Essentials of k-essence*. Physical Review D, **63**, N10, 3510 (2010). [arXiv:astro-ph/0006373]
- [13] Myrzakulov R. *Fermionic K-essence*. [arXiv:1011.4337]

- [14] Jamil M., Momeni D., Serikbayev N.S. and Myrzakulov R. *FRW and Bianchi type I cosmology of f-essence*. Astrophysics and Space Science, **339**, 37 (2012). [arXiv:1112.4472]
- [15] Tsyba P., Yerzhanov K., Esmakhanova K., Kulnazarov I., Nugmanova G., Myrzakulov R. *Reconstruction of f-essence and fermionic Chaplygin gas models of dark energy*. [arXiv:1103.5918]
- [16] Jamil M., Myrzakulov Y., Razina O., Myrzakulov R. *Modified Chaplygin gas and solvable f-essence cosmologies*. Astrophysics and Space Science, **336**, 315-325 (2011). [arXiv:1107.1008]
- [17] Kulnazarov I., Yerzhanov K., Razina O., Myrzakul S., Tsyba P. and Myrzakulov R. *G-essence with Yukawa Interactions*. European Physical Journal C, **71**, 1698 (2011). [arXiv:1012.4669]
- [18] Bamba K., Razina O., Yerzhanov K., Myrzakulov R. *Cosmological evolution of equation of state for dark energy in G-essence models*. International Journal of Modern Physics D. **22** 1350023 (2013). [arXiv:1203.2804]
- [19] Razina O., Kulnazarov I., Yerzhanov K., Tsyba P., Myrzakulov R. *Einstein-Cartan gravity with scalar-fermion interactions*. Central European Journal of Physics, **10**, N1, 47-50 (2012). [arXiv:1012.5690]
- [20] Yerzhanov K.K., Tsyba P.Yu., Myrzakul Sh.R., Kulnazarov I.I., Myrzakulov R. *Accelerated expansion of the Universe driven by G-essence*. [arXiv:1012.3031]
- [21] Buchdahl H.A. *Non-linear Lagrangians and cosmological theory*. Monthly Notices of the Royal Astronomical Society, **150**, 1-8 (1970).
- [22] Nojiri S., Odintsov S.D., Saez-Gomez D. *Cosmological reconstruction of realistic modified  $F(R)$  gravities*. Physics Letters B, **681**, N74, 74-80 (2009). [arXiv:0908.1269]
- [23] Elizalde E., Nojiri S., Odintsov S.D., Saez-Gomez. *Unifying inflation with dark energy in modified  $F(R)$  Horava-Lifshitz gravity*. European Physical Journal C, **70**, N1-2, 351 (2010). [arXiv:1006.3387]
- [24] Bengochea G.R. and Ferraro R. *Dark torsion as the cosmic speed-up*. Physical Review D, **79**, 124019 (2009). [arXiv:0812.1205]
- [25] Linder E.V. *Einstein's Other Gravity and the Acceleration of the Universe*. Physical Review D, **81**, 127301 (2010). [arXiv:1005.3039]
- [26] Myrzakulov R. *Accelerating universe from  $F(T)$  gravity*. The European Physical Journal C, **71**, 1752 (2011). [arXiv:1006.1120]
- [27] Yerzhanov K.K., Myrzakul Sh.R., Kulnazarov I.I., Myrzakulov R. *Accelerating cosmology in  $F(T)$  gravity with scalar field*. [arXiv:1006.3879]
- [28] Jamil M., Momeni D., Myrzakulov R. *Attractor Solutions in  $f(T)$  Cosmology*. The European Physical Journal C, **72**, 1959 (2012). [arXiv:1202.4926]
- [29] Bamba K., Jamil M., Momeni D., Myrzakulov R. *Generalized second law of thermodynamics in  $f(T)$  gravity with entropy corrections*. Astrophysics and Space Science, **344**, N1, 259-267 (2012). [arXiv:1202.6114]
- [30] Rodrigues M.E., Hamani Daouda M., Houndjo M.J.S., Myrzakulov R., Sharif M. *Inhomogeneous universe in  $f(T)$  theory*. Gravitation and Cosmology, **20**, N2, 80-89 (2014). [arXiv:1205.0565]
- [31] Jamil M., Yesmakhanova K., Momeni D., Myrzakulov R. *Phase space analysis of interacting dark energy in  $f(T)$  cosmology*. Central European Journal of Physics, **10**, 1065-1071 (2012). [arXiv:1207.2735]

- [32] Jamil M., Momeni D., Myrzakulov R. *Stability of a non-minimally conformally coupled scalar field in  $F(T)$  cosmology*. The European Physical Journal C, **72**, 2075 (2012). [arXiv:1208.0025]
- [33] Jamil M., Momeni D., Myrzakulov R. *Resolution of dark matter problem in  $f(T)$  gravity*. The European Physical Journal C, **72**, 2122 (2012). [arXiv:1209.1298]
- [34] Jamil M., Momeni D., Myrzakulov R., Rudra P. *Statefinder Analysis of  $f(T)$  Cosmology*. Journal of the Physical Society of Japan, **81**, N11, 114004 (2012). [arXiv:1211.0018]
- [35] Jamil M., Momeni D., Myrzakulov R. *Energy conditions in generalized teleparallel gravity models*. General Relativity and Gravitation, **45**, N1, 263-273 (2013). [arXiv:1211.3740]
- [36] Myrzakulov R. *Cosmology of  $F(T)$  gravity and  $k$ -essence*. Entropy, **14**, N9, 1627-1651 (2012). [arXiv:1212.2155]
- [37] Jamil M., Momeni D., Myrzakulov R. *Wormholes in a viable  $f(T)$  gravity*. The European Physical Journal C, **73**, 2267 (2013). [arXiv:1212.6017]
- [38] Aslam A., Jamil M., Momeni D., Myrzakulov R. *Noether Gauge Symmetry of Modified Teleparallel Gravity Minimally Coupled with a Canonical Scalar Field*. Canadian Journal of Physics, **91**, N1, 93-97 (2013). [arXiv:1212.6022]
- [39] Rodrigues M.E., Houndjo M.J.S., Momeni D., Myrzakulov R. *Planar Symmetry in  $f(T)$  gravity*. International Journal of Modern Physics D, **22**, N8, 1350043 (2013). [arXiv:1302.4372]
- [40] Houndjo M.J.S., Momeni D., Myrzakulov R., Rodrigues M.E. *Evaporation phenomena in  $f(T)$  gravity*. Canadian Journal of Physics, **93**, N4, 377-383 (2015). [arXiv:1304.1147]
- [41] Farooq M.Umar, Jamil M., Momeni D., Myrzakulov R. *Reconstruction of  $f(T)$  and  $f(R)$  gravity according to  $(m,n)$ -type holographic dark energy*. Canadian Journal of Physics, **91**, 703-708 (2013). [arXiv:1306.1637]
- [42] Rodrigues M.E., Houndjo M.J.S., Tossa J., Momeni D., Myrzakulov R. *Charged Black Holes in Generalized Teleparallel Gravity*. Journal of Cosmology and Astroparticle Physics, **1311**, 024 (2013). [arXiv:1306.2280]
- [43] Aslam A., Jamil M., Myrzakulov R. *Noether gauge symmetry for the Bianchi type I model in  $f(T)$  gravity*. Physica Scripta, **88**, 025003 (2013). [arXiv:1308.0325]
- [44] Jamil M., Momeni D., Myrzakulov R. *Warm Intermediate Inflation in  $F(T)$  Gravity*. International Journal of Theoretical Physics, **54**, 1098-1112 (2015). [arXiv:1309.3269]
- [45] Myrzakulov R., Saez-Gomez D., Tsyba P. *Cosmological solutions in  $F(T)$  gravity with the presence of spinor fields*. International Journal of Geometric in Modern Physics, **12**, N02, 1550023 (2014). [arXiv:1311.5261]
- [46] Gudekli E., Myrzakul A., Myrzakulov R. *Teleparallelism by inhomogeneous dark fluid*. Astrophysics and Space Science, **359**, N2, 64 (2015). [arXiv:1312.1112]
- [47] Momeni D., Myrzakulov R. *Conformal Invariant Teleparallel Cosmology*. European Physical Journal Plus, **129**, 137 (2014). [arXiv:1404.0778]
- [48] Momeni D., Myrzakulov R. *Cosmological reconstruction of  $f(T, \mathcal{T})$  Gravity*. International Journal of Geometric Methods in Modern Physics, **11**, 1450077 (2014). [arXiv:1405.5863]
- [49] Chattopadhyay S., Jawad A., Momeni D., Myrzakulov R. *Pilgrim Dark Energy in  $f(T, T_G)$  cosmology*. Astrophysics and Space Science, **353**, N1, 279-292 (2014). [arXiv:1406.2307]
- [50] Momeni D., Myrzakulov R. *On existence of a possible Lorentz invariant modified gravity in Weitzenböck spacetime*. Astrophysics and Space Science, **360**, N28 (2015). [arXiv:1511.01205]

- [51] Jawad A., Momeni D., Rani S., Myrzakulov R. *Dynamical Instability of Cylindrically Symmetric Collapsing Star in Generalized Teleparallel Gravity*. [arXiv:1511.03655]
- [52] Harko T., Lobo F.S.N., Nojiri S. and Odintsov S.D.  *$f(R, T)$  gravity*, Physical Review D, **84**, 024020 (2011). [arXiv:1104.2669]
- [53] Jamil M., Momeni D., Raza M. and Myrzakulov R. *Reconstruction of some cosmological models in  $f(R, T)$  gravity*. European Physical Journal C, **72**, 1999 (2012). [arXiv:1107.5807]
- [54] Sharif M., Rani S., Myrzakulov R. *Analysis of  $F(R, T)$  Gravity Models Through Energy Conditions*. European Physical Journal Plus, **128**, N11, 123 (2013). [arXiv:1210.2714]
- [55] Myrzakulov R. *Dark Energy in  $F(R, T)$  Gravity*. [arXiv:1205.5266]
- [56] Myrzakulov R. *FRW Cosmology in  $F(R, T)$  gravity*. The European Physical Journal C, **72**, N11, 2203 (2012). [arXiv:1207.1039]
- [57] Houndjo M.J.S., Rodrigues M.E., Mazhari N.S., Momeni D. and Myrzakulov R. *Higher-Derivative  $f(R, \square R, T)$  Theories of Gravity*. [arXiv:1601.00196]
- [58] Capozziello S., De Laurentis M. and Odintsov S.D. *Hamiltonian dynamics and Noether symmetries in Extended Gravity Cosmology*. The European Physical Journal C, **72**, 2068 (2012).
- [59] Capozziello S. and Lambiase G. *Selection Rules in Minisuperspace Quantum Cosmology*. General Relativity and Gravitation, **32**, 673-696 (2000). [arXiv:gr-qc/9912083]
- [60] Capozziello S. and de Ritis R. *Noether's symmetries and exact solutions in flat nonminimally coupled cosmological models*. Classical and Quantum Gravity, **11**, 107 (1994).
- [61] Capozziello S. and de Ritis R. *Nonminimal coupling and matter cosmologies*. Physics Letters A, **195**, 48-52 (1994).
- [62] Capozziello S. and de Ritis R. *String dilaton cosmology with exponential potential*. Physics Letters A, **177**, 1 (1993).
- [63] De Souza R.C., Kremer G.M. *Noether symmetry for non-minimally coupled fermion fields*. Classical and Quantum Gravity, **25**, N22, 225006 (2008). [arXiv:0807.1965]
- [64] Jamil M., Mahomed F.M. and Momeni D. *Noether symmetry approach in  $f(R)$ -tachyon model*. Physics Letters B, **702**, 315 (2011). [arXiv:1105.2610]
- [65] Wei H., Guo X.-J., Wang L.F. *Noether Symmetry in  $F(T)$  Theory*. Physics Letters B, **707**, N2, 298 (2012). [arXiv:1112.2270]
- [66] Atazadeh K., Darabi F.  *$f(T)$  cosmology via Noether symmetry*. The European Physical Journal C, **72**, N5, 2016 (2012). [arXiv:1112.2824]
- [67] Mohseni Sadjadi H. *Generalized Noether symmetry in  $F(T)$  gravity*. Physics Letters B, **718**, N2, 270 (2012). [arXiv:1210.0937]
- [68] Dong H., Wang J. Meng X. *The distinctions between Lambda CDM and  $F(T)$  gravity according Noether symmetry*. The European Physical Journal C, **73**, N8, 2543 (2013). [arXiv:1304.6587]
- [69] Basilakos S., Capozziello S., De Laurentis M., Paliathanasis A., Tsamparlis M. *Noether symmetries and analytical solutions in  $F(T)$ -cosmology: A complete study*. Physical Review D, **88**, 103526 (2013). [arXiv:1311.2173]
- [70] Paliathanasis A., Basilakos S., Saridakis E.N., Capozziello S., Atazadeh K., Darabi F., Tsamparlis M. *New Schwarzschild-like solutions in  $f(T)$  gravity through Noether symmetries*. Physical Review D, **89**, 104042 (2014). [arXiv:1402.5935]

- [71] Kucukakca Y. *Scalar Tensor Teleparallel Dark Gravity via Noether Symmetry*. The European Physical Journal C, **73**, N2, 7 (2013). [arXiv:1404.7315]
- [72] Kucukakca Y. *Teleparallel dark energy model with a fermionic field via Noether symmetry*. The European Physical Journal C, **74**, N10, 3086 (2014). [arXiv:1407.1188]
- [73] Gecim G., Kucukakca Y., Sucu Y. *Noether gauge symmetry of Dirac field in 2+1 dimensional gravity*. Advances in High Energy Physics, **2015**, 567395 (2015). [arXiv:1410.5283]
- [74] Jamil M., Momeni D., Myrzakulov R. *Noether symmetry of  $F(T)$  cosmology with quintessence and phantom scalar fields*. The European Physical Journal C, **72**, 2137 (2012). [arXiv:1210.0001]
- [75] Momeni D., Myrzakulov R., Gudekli E. *Cosmological viable Mimetic  $f(R)$  and  $f(R, T)$  theories via Noether symmetry*. International Journal of Geometric Methods in Modern Physics, **12**, 1550101 (2015). [arXiv:1502.00977]
- [76] Momeni D., Myrzakulov R. *Noether symmetry in Horndeski Lagrangian*. [arXiv:1410.1520]
- [77] Capozziello S., De Laurentis M. and Myrzakulov R. *Noether Symmetry Approach for teleparallel-curvature cosmology*. International Journal of Geometric Methods in Modern Physics, **12**, N09, 1550095 (2015). [arXiv:1412.1471]
- [78] Paliathanasis A., Basilakos S., Saridakis E.N., Capozziello S., Atazadeh K., Darabi F., Tsampanlis M. *New Schwarzschild-like solutions in  $f(T)$  gravity through Noether symmetries* Physical Review D, **89**, 104042 (2014). [arXiv:1402.5935]
- [79] Basilakos S., Capozziello S., De Laurentis M., Paliathanasis A., Tsampanlis M., *Noether symmetries and analytical solutions in  $f(T)$ -cosmology: A complete study*. Physical Review D, **88**, 103526 (2013). [arXiv:1311.2173]